New Mathematical Models of the Generalized Vehicle Routing Problem and Extensions

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INTRODUCTION

Let

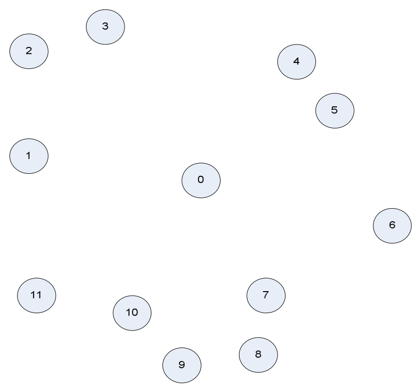
G = (V, A) be a directed graph with

V = { 0, 1, 2, ... ,n } as the set of vertices and

A = { (i, j) : $i, j \in V$, $i \neq j$ } as the set of arcs.

- Node 0 represents the depot and remaining n nodes represent geographically dispersed customers.

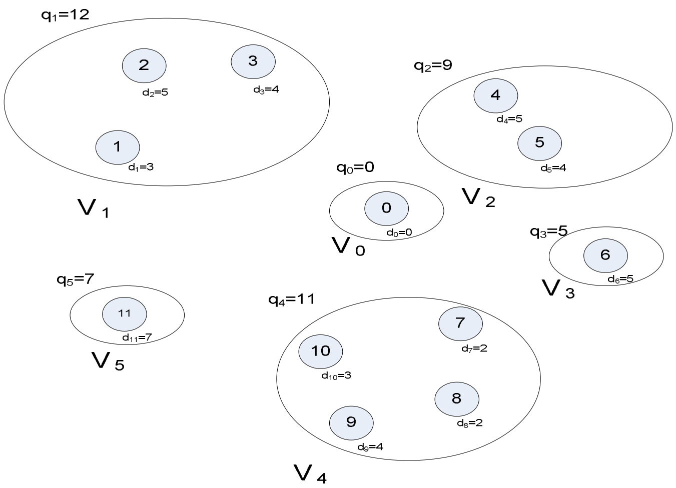
Network of 11 customers



INTRODUCTION

- The node set V is clustered into k mutually exclusive nonempty subsets Vi such that n$V = V_0$, V_1 ,....... , V_k, where V_0 = {0} is the depot (origin).
- Each customer has a certain amount of demand and the total demand of each cluster can be satisfied via any of its nodes.
- There exist m identical vehicles, each with a capacity Q.
- There also exists a nonnegative cost c_{ij} associated with each arc (i, j)A.

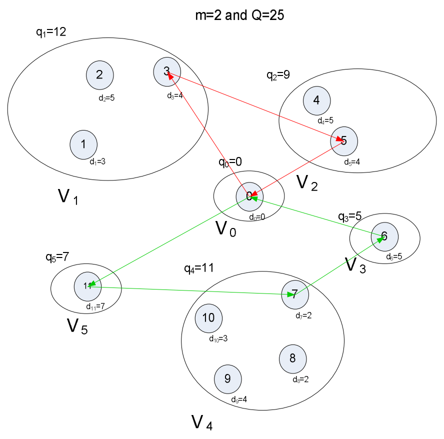
Generalized VRP



GENERALIZED VEHICLE ROUTING PROBLEM

The GVRP consists of finding minimum total cost tours of *m* vehicles starting and ending at the depot, such that each cluster should be visited by exactly one vehicle at any of its nodes, the entering and leaving nodes of each cluster is the same and the load of each vehicle does not exceed its capacity *Q*.

Feasible solution of GVRP



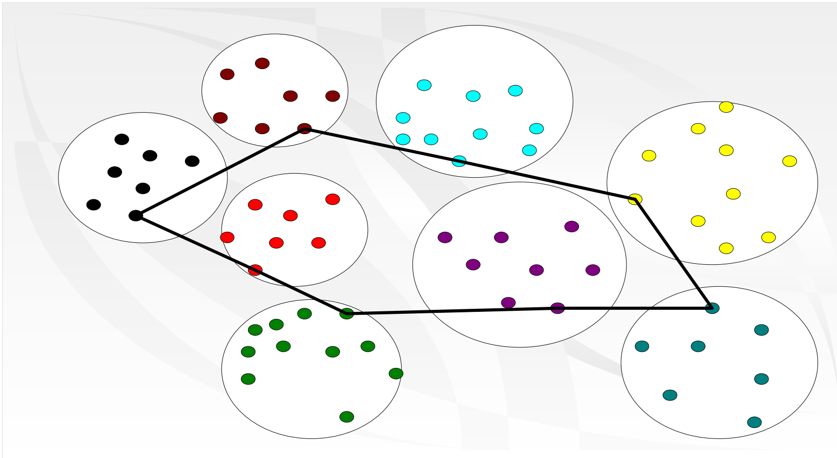
GENERALIZED TSP

- GVRP is also a variant of the Generalized Traveling Salesman Problem (GTSP), which is an extension of the well known Traveling Salesman Problem.
- An extensive research exists on the GTSP (see for example, Laporte and Nobert (1983); Noon and Bean(1991, 1993); Dimitrijevic and Saric (1997); Fischetti et al. (1995, 2002); and Ben-Arieh et al. (2003)).
- Integer linear programming formulations for GTSP are presented by Laporte and Nobert(1983) and Fischetti et al. (1995,2002). In these formulations, the number of the constraints grow exponentially with the number of the nodes of the graph.

GENERALIZED TSP

- Other research mentioned above focus on the transformation of the GTSP into the TSP. Recently, Pop (2007) proposed six new integer programming formulations four of them are polynomial size formulations for GTSP.
- We could not observe any formulation for the multiple traveler case of the GTSP, namely the Generalized Multiple Traveling Salesman Problem (GmTSP).
- In fact, the GVRP can be considered as an extension of the GmTSP where travelers are turn to be vehicles having limited capacities and clusters have a demand to be satisfied.

GENERALIZED TSP



SOME APPLICATONS

The GVRP and its special cases may arise in real-life applications such as

- loop material flow design,
- post-box collection,
- arc routing,
- computer operations,
- manufacturing, logistics, and
- distribution of goods by sea to a potential number of harbors (see Laporte et al. (1996); Ghiani and Improta(2000)).

PREVIOUS SDUDIES

- The GVRP has been introduced and a solution approach is presented by Ghiani and Improta (2000).
- To the best knowledge of the authors, this is the only solution approach for the GVRP, where a transformation of the GVRP into a Capacitated Arc Routing Problem is presented.
- Kara and Bektaş (2003) proposed an integer programming formulation for GVRP with polynomially increasing binary variables and constraints.

GVRP NOTATIONS

Decision Variables:

Define,

$$x_{ij} = \begin{cases} 1, \text{if arc } (i, j) \text{ is on the tour of any vehicle}, & i \in V_p, \quad j \in V_r, \quad p \neq r, \quad p, r \in K \\ 0, \text{otherwise} \end{cases}$$

$$w_{pr} = \begin{cases} 1, \text{if there is a path from cluster } p \text{ to cluster } r, \ p, r \in K \\ 0, \text{otherwise} \end{cases}$$

GVRP NOTATIONS

Parameters:

Let,

c_{ij} : cost of traveling from node i to node j, $i \neq j,\ i \in V_p,\ j \in V_r,\quad p \neq r,\ \ p, r \in K$

d_i : demand of customer i, $\quad i = 1, 2, \ldots, n$

q_r : demand of cluster r, $q_r = \sum_{i \in V_r} d_i$, $\quad r \varepsilon K$

m : number of vehicles (tours),

Q : capacity of each vehicle.

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Cluster Degree Constraints

- For each cluster excluding V_0, there can only be a single outgoing arc to any other node belonging to other clusters. This is implied by the following constraints:

$$\sum_{i \in V_p} \sum_{j \in V \setminus V_p} x_{ij} = 1 \qquad , p \neq 0, p \in K \qquad (1)$$

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Cluster Degree Constraints

- There can only be a single incoming (entering) arc to a cluster from any other node belonging to other clusters, excluding V_0. This is implied by the following constraints:

$$\sum_{i \in V \setminus V_p} \sum_{j \in V_p} x_{ij} = 1 \qquad , p \neq 0, p \in K \qquad (2)$$

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DEPOT IN/OUT

- There should be m leaving arcs from and m entering arcs to the home city (origin), which are implied by

$$\sum_{i=1}^{n} x_{0i} = m \qquad (3)$$

$$\sum_{i=1}^{n} x_{i0} = m \qquad (4)$$

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Cluster Connectivity Constraints

- The entering and leaving nodes should be the same for each cluster, which is satisfied by

$$\sum_{i \in V \setminus V_p} x_{ij} - \sum_{i \in V \setminus V_p} x_{ji} = 0, \qquad j \in V_p, \qquad p \in K \qquad (5)$$

- Flows from cluster *p* to cluster *r* are defined by w_{pr}. Thus, w_{pr} should be equal to the sum of x_{ij}'s from V_p to V_r. Hence, we write

$$w_{pr} = \sum_{i \in V_p} \sum_{j \in V_r} x_{ij}, \qquad p \neq r, \qquad p, r \in K \qquad (6)$$

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GENERAL FORMULATION

$$\text{Minimize} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to: (1),(2),(3),(4),(5),(6) and

+ Capacity Bounding constraints, (7)

+ Subtour elimination constraints, (8)

$$x_{ij} \in \{0,1\}, \forall (i, j) \in A, \qquad (9)$$

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NODE BASED FORMULATION

- In addition to the decision variables defined before, let's define the following auxiliary variables.

  u_p : Load of a vehicle just after leaving cluster p (collection case) or delivered amount of the goods from a vehicle just after leaving p (delivery case), $p \neq 0$, $p \in K$.

NODE BASED FORMULATION

- **Proposition 1**: The following inequalities are valid capacity bounding constraints for GVRP.

$$u_p - \sum_{\substack{s \in K \\ s \neq p}} q_s w_{sp} \geq q_p, \qquad p \neq 0, p \in K \qquad (10)$$

$$u_p - (Q - q_p) w_{0p} \leq Q, \qquad p \neq 0, p \in K \qquad (11)$$

where $q_0 = 0$.

NODE BASED FORMULATION
Subtour elimination constraints

- Formation of any subtour between clusters excluding V_0 will not allowed by the following constraint:

$$u_p - u_r + Qw_{pr} + (Q - q_p - q_r)w_{rp} \le Q - q_r, \quad p \ne r \ne 0, \quad p, r \in K \qquad (12)$$

where $w_{pr} = 0$ whenever $q_p + q_r > Q$, $p \neq r \in K$.

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NODE BASED FORMULATION

- The first integer linear programming formulation of the GVRP is given by:

$$\textbf{F(1)}: \text{Minimize} \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \quad \text{subject to } (1)-(6) \text{ and } (9)-(12)$$

where, x_{ij}=0 whenever i,j ε V_r, rεK and w_{pr}=0 whenever q_p+q_r > Q.

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NODE BASED FORMULATION

- F(1) is structurally similar to Kara-Bektaş(2003) formulation,
- We show below that F(1) produces stronger lower bound than Kara-Bektaş formulation.
- **Proposition 2:** Let linear programming relaxation of Kara-Bektaş formulation and F(1) are shown as LPR(K-B) and LPR(F1), respectively,then LPR(F1) $\geq$ LPR(K-B).

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FLOW BASED FORMULATION

- Let's define other auxiliary decision variables as:
- y_{pr} : is the amount of goods picked up (or delivered in the case of delivery) on the route of a vehicle just after leaving p^{th} cluster if a vehicle goes from cluster *p* to cluster *r;* zero otherwise.

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FLOW BASED FORMULATION

- **Proposition 3:** The following relations are valid bounding and subtour elimination constraints for GVRP.

$$y_{rp} \le (Q - q_p) w_{rp} \quad , \quad r \ne p \, , \quad r, p \in K \qquad (13)$$

$$y_{rp} \ge q_r w_{rp} \quad , \quad r \ne p \, , \quad r, p \in K \qquad (14)$$

$$\sum_{p=1}^{k} y_{p0} = \sum_{p=1}^{k} q_p \quad , \quad (p \in K) \qquad (15)$$

$$\sum_{p=1}^{k} y_{rp} - \sum_{p=1}^{k} y_{pr} = q_r \quad , \quad p, r \in K, \text{ for all } r \ne 0 \qquad (16)$$

where $y_{0p} = 0$, *for* all $p \in K$, and $q_0 = 0$.

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FLOW BASED FORMULATION

F(2): Minimize $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$ subject to (1) - (6), (9), (13) – (16)

where, x_{ij}=0 whenever i,j ε V_r , r ε K and w_{pr}=0 whenever $q_p + q_r > Q$.

In addition, $y_{0p} = 0$ *for* all $p \in K$, and q_0=0. Note that, constraints given in (14) guarantee that, $y_{pr} \ge 0$ for all $p \ne r$, $p, r \in K$

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SPECIAL CASES

- In GVRP, let the total demand of each cluster equal to 1 and there is no capacity restriction for the vehicle. In this special case, GVRP may be named as the Generalized Multiple Traveling Salesman Problem (GmTSP).
- For GmTSP, the meaning of auxiliary variables u_i's and y_{rp}'s and parameters of the model will be as follows:
- u_p : The rank order of cluster p on the tour of a vehicle (visit number), p ε K.
- y_{pr} : is the total number of the arc on the route of a vehicle traveled just after leaving p^{th} cluster if a vehicle goes from cluster *p* to cluster *r;* zero otherwise.

GmTSP/ GTSP

- q_p = 1, for all pεK and q_0=0,
- Q = k-m+1 is the maximum number of clusters that a vehicle can visit.
- The GmTSP reduces to the Generalized TSP (GTSP) when m = 1, i.e., when there is a single traveler.
- With those above, F(1) and F(2) turn out formulations of GmTSP(GTSP)

Clustered Vehicle Routing Problem(CGVRP)

- Let us define extension of GVRP as Clustered Vehicle Routing Problem (CGVRP) where
- all the nodes of each cluster must be on a route of a vehicle consecutively.
- Thus, for the case of CGVRP, the problem is to find a least cost tours of *m* vehicles starting and ending at the depot such that,

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Clustered Vehicle Routing Problem(CGVRP)

- In CGVRP, each node of the entire graph is visited exactly once by performing a Hamiltonian path within each cluster, and
- Each cluster should be visited by exactly one vehicle at any of its nodes, and
- The load of each vehicle does not exceed its capacity *Q*.

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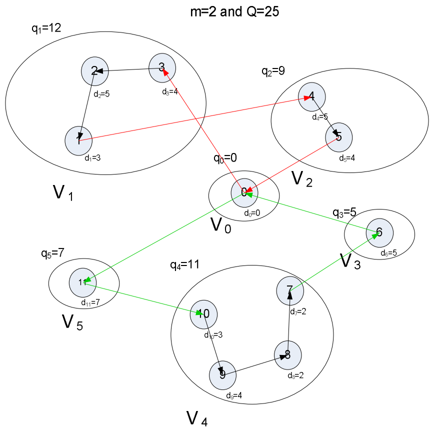
Clustered VRP



Degree constraints of CGVRP

we add the following node degree constraints to the both formulations and omit connectivity constraints given in (5)

$$\sum_{i=0}^{n} x_{ij} = 1, \qquad j = 1,2,\ldots,n$$

$$\sum_{j=0}^{n} x_{ij} = 1, \qquad i = 1,2,\ldots,n$$

Subtour Elimination Constraints of CGVRP

v_i: The rank order of node $_i$ on the tour of a vehicle (visit number).

$$v_i - v_j + |V_p| x_{ij} + (|V_p| - 2) x_{ji} \le |V_p| - 1, \qquad p \in K, \quad i, j \in V_p,$$

$$v_i \ge 1, \qquad i \in V_p, \; p \in K,$$

Subtour Elimination Constraints of CGVRP

t_{ij}: is the number of the arc traveled within the given cluster on the route of a vehicle just after leaving i th node if a vehicle goes from node i to node j ; zero otherwise

$$\sum_{j=0}^{n} t_{ij} - \sum_{j=0}^{n} t_{ji} = 1, \qquad \forall i \in V_p, p \in K,$$

$$t_{ij} \le (|V_k| - 1) x_{ij}, \qquad (i, j) \in V_k, \; \mathrm{k} \in \mathrm{K}$$

Conclusions and Further Research Area

- Two polynomial size formulations for the generalized vehicle routing problem are presented.
- Generalized multiple traveling salesman problem is defined and, as a special case, it is shown that both of the proposed formulation to the GVRP reduce to the formulation of the mGTSP.
- GVRP is extended where each tour of a vehicle continue in each cluster and this extension is defined as clustered vehicle routing problem and formulations proposed for GVRP are adapted for CVRP easily.
- Computational comparations of the proposed formulations and or to develop exact algorithms for GVRP and CGVRP seem further research areas.

THANKS